CYCLIC QUADRILATERALS

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Cyclic quadrilaterals, also known as chordal quadrilaterals, are quadrilaterals whose vertices can all be inscribed on the same circle[4]. This classification is notable because of specific properties these quadrilaterals possess which non-cyclic ones do not. Most of these properties hold for convex cyclic quadrilaterals only, but non-convex (i.e. crossed) quadrilaterals do exist. Additionally, certain subsets of the quadrilaterals are cyclic by construction, such as squares and rectangles. Likewise, other subsets of the quadrilaterals can never be cyclic, such as any rhombus which is not a square. Euclid himself provided one of the earliest references to these quadrilaterals in Book 3, Proposition 22 of his *Elements*[1]. A number of small theorems concern cyclic quadrilaterals in particular, such as Ptolemy's Theorem, the Intersecting Chords Theorem, and the Japanese Theorem.

As for specific definitions, the following statements are equivalent:

(1) A convex quadrilateral is cyclic.
(2) The opposite angles of a quadrilateral are supplementary[2] (a direct consequence of Prop. 22).
(3) The perpendicular bisectors of the sides of a quadrilateral intersect at exactly one point.
(4) The product of the lengths of the diagonals is equivalent to the sum of the products of the opposite sides of the quadrilateral[5]. This is also known as Ptolemy's Theorem.
(5) The tangent of half of any angle of the quadrilateral times the tangent of half of its opposite angle is equal to one[3].

Ptolemy was not specifically concerned with studying cyclic quadrilaterals. Instead, he developed his theorem to assist in the creation of his table of chords, a trigonometric table relating to the value of sines. As stated in definition 4, a quadrilateral is cyclic if and only if the products of its diagonals' lengths are equivalent to the sum of the products of the opposite sides of the quadrilateral. Furthermore, the converse is true as well, so if the equivalence holds for a given quadrilateral, then that quadrilateral can be inscribed in a circle.

[1]Prop 22: The sum of the opposite angles of quadrilaterals in circles equals two right angles.

Ptolemy's Theorem is really an extension of the Pythagorean Theorem: if the quadrilateral inscribed in a circle is a rectangle, then applying the Pythagorean Theorem is equivalent to applying Ptolemy's Theorem, both of which just imply that the diagonals of the quadrilateral intersect at the origin of the circle.

Ptolemy's Theorem has two basic proofs, one of which relies of the similarity of triangles and one that relies on trigonometric identities. However, since the measure of certain angles are warped in non-Euclidean spaces, the angular proof fails and thus Ptolemy's Theorem does not necessarily hold in neutral geometry. The proof of Ptolemy's Theorem by similarity of triangles is as follows:

(1) Let $\square ABCD$ be a cyclic quadrilateral. By construction, $\angle BAC = \angle BCD$ and $\angle ADB = \angle ACB$.
(2) Construct point X on chord $\overline{AC}$ such that $\angle ABX = CBD$.
(3) By common angles, $\triangle ABK \sim \triangle DBC$ and $\triangle ABD \sim \triangle KBC$.
(4) Thus, $|\overline{AX}|/|\overline{AB}| = |\overline{CD}|/|\overline{BD}|$ and $|\overline{CX}|/|\overline{BC}| = |\overline{DA}|/|\overline{BD}|$. Likewise, $|\overline{AX}|/|\overline{BD}| = |\overline{AB}|/|\overline{CD}|$ and $|\overline{CX}|/|\overline{BD}| = |\overline{BC}|/|\overline{DA}|$.
(5) Adding the two equalities yields $|\overline{AX}| * |\overline{BD}| + |\overline{CX}| * |\overline{BD}| = |\overline{AB}| * |\overline{CD}| + |\overline{BC}| * |\overline{DA}|$.
(6) Factorizing, $(|\overline{AX}| + |\overline{CX}|) * |\overline{BD}| = |\overline{AB}| * |\overline{CD}| + |\overline{BC}| * |\overline{DA}|$.
(7) Since $AX + XC = AC$, $|\overline{AC}| * |\overline{BD}| = |\overline{AB}| * |\overline{CD}| + |\overline{BC}| * |\overline{DA}|$. This proves the theorem.

The Japanese Theorem is perhaps the strangest and most interesting of the listed theorems. It relates the incircles of four triangles formed by diagonals on the inside of a cyclic quadrilateral. Specifically, it states that, given any cyclic quadrilateral, the origins of the incircles[2] of its inner triangles form the vertices of a rectangle[1]. Note that the four triangles mentioned are not the ones formed by the diagonals themselves, but rather the triangles formed by removing a single vertex from the quadrilateral.

The proof is fairly straightforward:

(1) Let $\square ABCD$ be a cyclic quadrilateral.
(2) From $\triangle BCD$, we have $\angle BI_{BCD}C = 90° + \angle BDC/2$.
(3) Likewise, $\angle BI_{ABC}C = 90° + \angle BAC/2$ from $\triangle ABC$.
(4) $\square ABCD$ is cyclic, so $\angle BDC = \angle BAC$, and thus $\angle BI_{BCD}C = \angle BI_{ABC}C$.

[2]The *incircle* of a triangle is a circle inscribed inside of a triangle such that all three sides of the triangle are tangent to the circle.

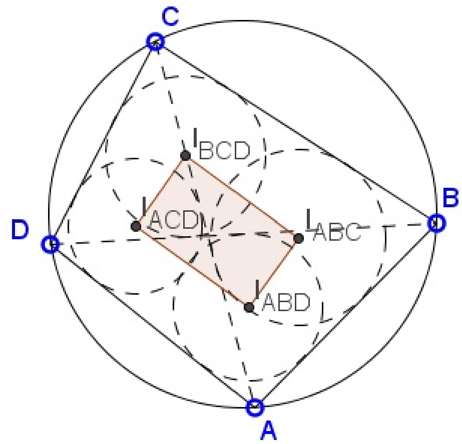

FIGURE 0.1. A cyclic quadrilateral demonstrating the Japanese Theorem.

(5) By Ptolemy's Theorem, $BI_{ABC}I_{BCD}C$ is therefore cyclic, so we have $\angle BCI_{BCD} + \angle BI_{ABC}I_{BCD} = 180°$ and $\angle BAI_{ABD} + \angle BI_{ABC}I_{ABD} = 180°$.

(6) $\angle BI_{ABC}I_{BCD} + \angle BI_{ABC}I_{ABD} = 360° - \angle BCD/2 - \angle BAD/2 = 270°$.

(7) $\angle I_{ABD}I_{ABC}I_{BCD} = 90°$. Repeating this method for the other three angles shows all four angles are right, and thus the quadrilateral formed is a rectangle.

Furthermore, this quadrilateral case of the Japanese Theorem may be extended to any cyclic polygon to show that the sum of the radii of all incircles formed by the subtriangles of the polygon is constant and independent of the way the triangles are formed. Since a cyclic quadrilateral may only be divided into triangles one of two ways, the case is fairly trivial, but the radii sum of $I_{BCD} + I_{ABD}$ is indeed equivalent to the radii sum of $I_{ABC} + I_{ACD}$.

In neutral geometry, it is much more difficult to determine whether or not a quadrilateral is cyclic. While cyclic quadrilaterals certainly exist in neutral geometry (by taking a circle and forming quadrilateral from four points on the circle), not all statements for cyclic quadrilaterals hold as they do in Euclidean geometry. For instance, if there is a triangle with an angle sum of less than 180°, then no rectangles exist. The Japanese Theorem is then moot, the quadrilateral formed by the centers of the incircles cannot possibly be rectangular. Furthermore, statements 2, 4, and 5 do not hold in such a geometry. If a triangle with angle sum equivalent to 180° exists, then the geometry is semi-Euclidean and all five statements hold. In the case of triangles having an angle sum of greater than 180°, the statements 2, 4, and 5 fail to hold again. The reason for the fragility of these particular statements is their dependence on angle and length measurements, both of which are quantities that can become distorted in neutral geometry. Ptolemy's Theorem is doubly fragile, as both proofs rely either on triangle congruence axioms or angle measurement axioms, neither of which necessarily hold in neutral geometry. Incidence axioms still hold, which explains the resilience of statements 1 and 3. It is always possible to inscribe a quadrilateral inside of a circle regardless of distortion of the geometry, and even in neutral geometry, perpendicular bisectors will still intersect at exactly one point when the geometry is consistent.

References

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