GEOMETRY FROM A TAXICAB

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When measuring distance between two points in Euclidean space, the most common metric used is the *Euclidean metric*. In Euclidean n-space, the distance function $d(a, b)$, which computes the distance between two points a and b, is given by the following formula:

$$d(a,b) = d(b,a) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \ldots + (a_n - b_n)^2} = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

We see this is a valid metric over any Euclidean n-space as $d(a, b) = d(b, a)$, $d(a, a) = 0$, and $d(a, c) \leq d(a, b) + d(b, c)$. This function's construction is identical to that of the L^2 norm, indeed, it is also known as the L^2 distance. However, applying instead the L^1 norm to Euclidean 2-space gives rise to what is known as the *taxicab plane*[2].

The taxicab plane is an interesting take on defining distance across Euclidean space. Instead of being able to travel from point to point "as the crow flies" to measure distance, we must instead measure on a component-by-component basis. Thus, the distance between two points is the combined distance between all corresponding coordinates of the points. In Euclidean 2-space, this is simple, as we already have the coordinate grid defined by the Cartesian coordinate system. The metric on the taxicab plane is defined as follows:

$$d_T(a,b) = d_T(b,a) = |a_1 - b_1| + |a_2 - b_2|$$

The metric can also easily be extended to any dimension using the generalized formula $\sum_{i=1}^{n} |a_i - b_i|$. In simplest terms, this formula could accurately describe the distance a taxicab would have to travel from a to b as it zigzags down city blocks. Additionally, all possible paths are of equal length, as long as the taxicab only makes turns that bring it nearer to its destination, i.e. the difference between the coordinates of the taxicab and the destination must get smaller with every turn the cab makes. Despite the properties of points, lines,

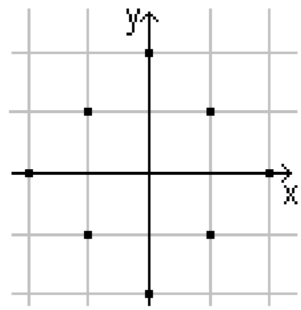

FIGURE 0.1. A circle of radius 2 in the taxicab plane.

and angles all not differing from Euclidean coordinate geometry, the odd notion of distance in the taxicab plane creates some interesting geometrical anomalies. For instance, a circle of radius r in the taxicab plane consists of the set of all points with distance r from the origin point of the circle. However, because our taxicab can't drive directly to these points but instead must make turns around the grid, the shape made by this construction ends up being a square, rotated 45° in standard Euclidean space[1].

Another notable quirk of the taxicab plane is that certain geometrical conditions of congruence fail us. In particular, the following are three "tests" for congruent triangles that do not work in taxicab space:

SAS: (Side-Angle-Side) Two triangles are congruent if they both have two pairs of sides that are of equal length and the angles between those side are of equal measurement.

ASA: (Angle-Side-Angle) Two triangles are congruent if they both have two pairs of angles that are of equal measurement and the sides between those angles are of equal length.

SSS: (Side-Side-Side) Two triangles are congruent if all three pairs of sides of the triangles are of equal length.

For each of these tests, we can construct two triangles which satisfy the test's requirements for congruence without actually being congruent, simply because of how distances are measured in the taxicab plane. In the SAS test, we can construct $\triangle ABC$ with $A(0,2)$, $B(0,0)$, and $C(2,0)$. This is a fairly simple isosceles triangle. We can then construct $\triangle DEF$ with $D(-1,1)$, $E(0,0)$, and $F(1,1)$. This, too, is a fairly simple isosceles triangle with a slightly smaller area than $\triangle ABC$. However, we can see that $\overline{AB} = \overline{DE}$, $\overline{BC} = \overline{EF}$, and both $\angle ABC$ and $\angle DEF$ are right angles. This satisfies the SAS test, as we have two pairs of sides that are the same length with the angles between them both being right angles. However, these triangles are clearly not congruent, as the area of $\triangle ABC$ is 2, while the area of $\triangle DEF$ is 1. One triangle is half the area of the other yet they share two congruent sides and angles!

Reusing $\triangle ABC$ from the previous example, we can define another triangle, $\triangle GHI$, with $G(-2,2)$, $H(0,0)$, and $I(2,2)$. Both of these triangles are isosceles, with a single angle being right, and the other two having measurements of half of a right angle. In addition to these three angles, the triangles share a pair of sides with equal length: $\overline{AC} = \overline{GI}$. Picking the two angles on both sides of these segments that correspond to each other, we have the requirements of the test satisfied. With $\angle BAC = \angle HGI$, $\angle BCA = \angle HIG$, and $\overline{AC} = \overline{GI}$, *ASA* implies that the two triangles are congruent. However, we once again know this to be false, as the area of $\triangle GHI$ is twice that of $\triangle ABC$.

Finally, we will construct two more triangles, $\triangle PQR$ and $\triangle XYZ$, to show the unreliability of the *SSS* test. If we define $P(-2,1)$, $Q(0,0)$, and $R(2,1)$, we have a $\triangle PQR$ with area 2. We can also define $X(2,2)$, $Y(2,-1)$, and $Z(0,0)$, to create $\triangle XYZ$ with area 3, so we know that $\triangle PQR \ncong \triangle XYZ$. However, we have $\overline{PQ} = \overline{XY}$, $\overline{QR} = \overline{YZ}$, and $\overline{PR} = \overline{XZ}$. All three pairs of sides have equal lengths, and yet the triangles are certainly not congruent. Clearly, we cannot rely on normal conventions of Euclidean space in the taxicab plane to always have the same outcome.

Despite these inconsistencies between standard Euclidean 2-space and the taxicab plane, some axioms are still satisfied in both.

- **C-1:** If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a unique point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.
- **C-2:** If $\overline{AB} \cong \overline{CD}$ and if $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. Moreoever, every segment is congruent to itself.
- **C-3:** If $A * B * C$, $A' * B' * C'$, $\overline{AB} \cong \overline{A'B'}$, and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.
- **C-4:** Given any $\measuredangle BAC$ and given any ray $\overrightarrow{A'B'}$ emanating from point A', then there is a unique ray $\overrightarrow{A'C'}$ on a given side of $\overleftrightarrow{A'B'}$ such that $\measuredangle B'A'C' \cong \measuredangle BAC$.
- **C-5:** If $\measuredangle A \cong \measuredangle B$ and $\measuredangle A \cong \measuredangle C$, then $\measuredangle B \cong \measuredangle C$. Moreover, every angle is congruent to itself.
- **C-6:** If two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of another triangle, then the two triangles are congruent.

Since the first axiom only relies on the notion of distance for congruence of lines, it holds in the taxicab plane as well. Even though distance is distorted along diagonals, we can negate the distortion by moving point B' closer to A'. Although the two segments would no longer be congruent in Euclidean 2-space, we would now have $AB \cong A'B'$ in taxicab space, so C-1 holds. Moreover, C-2 holds vacuously, as the taxicab metric is valid. All segments in the taxicab plane are thus congruent to themselves (reflexive) and the distance function is symmetric and transitive. We can derive C-3's veracity from C-2. Given $A * B * C$ and $A' * B' * C'$, we know $\overline{AB} \cap \overline{BC} = \overline{AC}$ and $\overline{A'B'} \cap \overline{B'C'} = \overline{A'C'}$. Replacing the segments in the construction of $\overline{AC}$ with their congruent counterparts, we have $\overline{A'B'} \cap \overline{B'C'} = \overline{AC}$. Thus, $\overline{AC} \cong \overline{A'C'}$. C-4 is another axiom which holds automatically. As angles are unaffected by the notion of distance in the taxicab plane, the axiom remains exactly the same and entirely intact. C-5 holds in the same fashion, and states that measures of angles in the taxicab plane are reflexive, symmetric, and transitive. C-6 is the only axiom that does not hold, as proven prior. Just as the angle axioms remain unaffected by the taxicab metric, so does the Angle Addition proposition 3.19.

References

[1] Martin Gardner. *The Last Recreations: Hydras, Eggs, and Other Mathematical Mystifications.* Basel; Boston; Berlin, 1997.

[2] Eugene F. Krause. *Taxicab Geometry: An Adventure in Non-Euclidean Geometry.* Dover Books on Mathematics. Dover Publications, 1975.